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On Varisolvency and Generalized Rational Approximation

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1. INTRODUCTION

A best approximation (F) to a function (f) from a varisolvent family, as defined by Rice, is characterized by alternation of $f - F$. We consider the space $C[a, b]$ of real valued continuous functions on an interval $[a, b]$, with the uniform norm. A best approximation to a function (f) from a family of generalized rational functions is also characterized by alternation criteria. However, only special cases of generalized rational approximating families are varisolvent.

We show that a family of generalized rational functions is varisolvent with respect to an extended definition of varisolvency introduced by Gillotte and McLaughlin. Some properties of families of generalized rational functions, in particular the alternation criteria, are then shown to follow from varisolvency. Further topics include a de La Vallée Poussin theorem, uniqueness results and approximation using a generalized weight function, for varisolvent families in the extended sense.

In [6], Gillotte and McLaughlin prove that the generalized exponential family E_n (cf. [12, p. 111]) is varisolvent in the extended sense. Thus, we note that with respect to Rice's definition, E_n and a family of generalized rational functions are not varisolvent except in special cases. However, both families are varisolvent with respect to the extended definition.

We shall use the following notation. For $g \in C[a, b]$, we define $\| g \| = \max_{x \in [a,b]} | g(x) |$. Further, given a nonempty family of functions $\mathscr{F}$ in $C[a, b]$, we say that $F \in \mathscr{F}$ is a best approximation to $f \in C[a, b]$ from $\mathscr{F}$ if $\| f - F \| \leqslant \| f - G \|$ for all G in $\mathscr{F}$.

2. RESULTS ON VARISOLVENCY

First, we present four definitions, which introduce the extended definition of varisolvency that appears in [6].

DEFINITION 1. Let $\{I_i\}_{i=1}^{n}$ be a sequence of closed intervals $(n \geqslant 1)$. The sequence will be called an increasing sequence of closed intervals if for every x in I_i and every y in I_{i+1} $(1 \leqslant i < n)$, $x < y$ is valid.

DEFINITION 2. Let $F \in C[a, b]$. Then F is said to have $n (n \geqslant 1)$ sign changes on $[a, b]$ if there exist points $\{x_i\}_{i=1}^{n+1}$, $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$, such that $F(x_i)\, F(x_{i+1}) < 0$ for all i $(1 \leqslant i \leqslant n)$.

DEFINITION 3. Let $\mathscr{F}$ be a family of functions in $C[a, b]$ and let $F \in \mathscr{F}$. The ordered pair of integers (n_1, n_2) with $n_1 \geqslant 0$ and $n_2 \geqslant 1$ is a degree of F with respect to $\mathscr{F}$ if the following conditions hold:

1. Let $\epsilon > 0$ and σ in $\{-1, 1\}$ be arbitrarily chosen. If $n_1 = 1$, there exists a $G \in \mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)(F(x) - G(x)) > 0$ on $[a, b]$. If $n_1 > 1$, if δ is an arbitrary element of $\{0, 1\}$ and if $\{[c_i, d_i]\}_{i=1}^{n_1-\delta}$ is an arbitrary increasing sequence of closed intervals, where $c_1 = a$ and $d_{n_1-\delta} = b$, then there exists a $G \in \mathscr{F}$ such that $\| F - G \| < \epsilon$ and $\sigma(-1)^i(F(x) - G(x)) > 0$ on $[c_i, d_i]$ for all i $(1 \leqslant i \leqslant n_1 - \delta)$.
2. If $G \in C[a, b]$ and $F(x) - G(x)$ has n_2 sign changes on $[a, b]$, then G is not in $\mathscr{F}$.

DEFINITION 4. Let $\mathscr{F}$ be a nonempty family of functions in $C[a, b]$. Then $\mathscr{F}$ will be called a varisolvent family if every $F \in \mathscr{F}$ has a degree with respect to $\mathscr{F}$.

In [6] it is shown that Definition 4 is an extension of Rice's definition of a varisolvent family. (Observe that part 2 of Definition 3 states that F has weak property Z of degree n_2 as defined by Dunham [5].)

For later use we introduce the following modification of Definitions 3 and 4.

DEFINITION 3′. Let $\mathscr{F}$ be a family of functions in $C[a, b]$ and let $F \in \mathscr{F}$. Then (n_1, n_2) will be a degree for F with respect to $\mathscr{F}$ if either:

1. $n_1 \geqslant 0$ and $n_2 \geqslant 1$ are integers and Definition 3 holds, or
2. $n_1 \geqslant 0$ is an integer, $n_2 = +\infty$ and part 1 of Definition 3 holds.

DEFINITION 4′. Let $\mathscr{F}$ be a nonempty family of functions in $C[a, b]$. Then $\mathscr{F}$ will be called a varisolvent family if every $F \in \mathscr{F}$ has a degree, according to Definition 3′, with respect to $\mathscr{F}$.

We observe that in the case $(n_1, +\infty)$ is a degree for $F \in \mathscr{F}$, only the integer n_1 gives information about the relation of F to the rest of the family $\mathscr{F}$. We also note that all theorems in [6] concerning varisolvent families, with respect to Definition 4, are valid with respect to Definition 4′. Unless specified otherwise, a "varisolvent family" will refer to a family satisfying Definition 4′.

We now present a number of new results about varisolvent families.

LEMMA 1. (de La Vallée Poussin). *Let $\mathscr{F}$ be a varisolvent family, let $F \in \mathscr{F}$ with degree (n_1, n_2), and let $f \in C[a, b]$. If there exist k points, $a \leqslant x_1 < \cdots < x_k \leqslant b$,* with *$k > n_2$ such that $(f(x_i) - F(x_i))(f(x_{i+1}) - F(x_{i+1})) < 0$ for all i $(1 \leqslant i \leqslant k-1)$ then it follows that*

$$\inf_{G \in \mathscr{F}} \| f - G \| \geqslant \min_{1 \leqslant i \leqslant k} | f(x_i) - F(x_i) |.$$

Proof. Assume not. It is easy to show that there exists a $G \in \mathscr{F}$ such that $F(x) - G(x)$ has n_2 sign changes on $[a, b]$. Q.E.D.

LEMMA 2. *Let $W(x, y)$ be a real valued function defined on $[a, b] \times (-\infty, \infty)$ satisfying* (a) *$W(x, y)$ is a strictly increasing function of y for every x in $[a, b]$, and* (b) *$W(x, y)$ is continuous on $[a, b] \times (-\infty, \infty)$. Let $\mathscr{F}$ be a varisolvent family on $[a, b]$. Then $\mathscr{W} = \{W(x, F(x)) \mid F \in \mathscr{F}, x \in [a, b]\}$ is a varisolvent family on $[a, b]$. Further, if $F \in \mathscr{F}$ has a degree (n_1, n_2), then (n_1, n_2) is also a degree for $W(x, F(x))$.*

(Note that Lemma 2 is a generalization of a result given by Kaufman and Belford in [7].)

Proof. $\mathscr{W}$ is a nonempty family of functions in $C[a, b]$. We show that each $W(x, F(x)) \in \mathscr{W}$ has a degree.

Let $W(x, F(x))$ be an arbitrary element of $\mathscr{W}$. Since $F \in \mathscr{F}$, it has a degree (n_1, n_2) with respect to $\mathscr{F}$. We show (n_1, n_2) is a degree for $W(x, F(x))$ with respect to $\mathscr{W}$.

Consider first n_2. If $n_2 = +\infty$, there is nothing to show. Thus, assume $1 \leqslant n_2 < \infty$ and assume there exists $W(x, G(x)) \in \mathscr{W}$ such that $W(x, F(x)) - W(x, G(x))$ has n_2 sign changes on $[a, b]$. Hence, there exist points $a \leqslant x_1 < \cdots < x_{n_2+1} \leqslant b$ with

$$[W(x_i, F(x_i)) - W(x_i, G(x_i))][W(x_{i+1}, F(x_{i+1})) - W(x_{i+1}, G(x_{i+1}))] < 0, \quad \text{for all } i \ (1 \leqslant i \leqslant n_2). \tag{1}$$

Recall that for any real number u, $\operatorname{sgn} u = u/|u|$ if $u \neq 0$ and $\operatorname{sgn} u = 0$ if $u = 0$. We observe that assumption (a) guarantees that

$$\operatorname{sgn}[W(x, F(x)) - W(x, G(x))] = \operatorname{sgn}[F(x) - G(x)], \tag{2}$$

for all $x \in [a, b]$ and any $G \in \mathscr{F}$. Applying (2) to (1), we obtain that $[F(x_i) - G(x_i)][F(x_{i+1}) - G(x_{i+1})] < 0$ for all i $(1 \leqslant i \leqslant n_2)$. This contradicts the fact that (n_1, n_2) is a degree for F.

Consider next n_1. Let $\epsilon > 0$ and σ in $\{-1, 1\}$ be given, and let $I = [\min_{x\in[a,b]} F(x) - \epsilon,\ \max_{x\in[a,b]} F(x) + \epsilon]$. Since $W(x, y)$ is uniformly continuous on $[a, b] \times I$, assumption (b) guarantees that an $\epsilon^* > 0$ exists such that $| z_1 - z_2 | < \epsilon^*$ implies $| W(x, z_1) - W(x, z_2)| < \epsilon$ for all $x \in [a, b]$ and all z_1, z_2 in I. Hence, for any $G \in \mathscr{F}$,

$$\| F - G \| < \epsilon^* \Rightarrow \| W(\cdot, F) - W(\cdot, G)\| < \epsilon. \tag{3}$$

Assume now that $n_1 = 1$. Since $F \in \mathscr{F}$ has (n_1, n_2) as a degree in $\mathscr{F}$, there exists a $G \in \mathscr{F}$ with $\| F - G \| < \epsilon^*$ and with $\sigma(-1)[F(x) - G(x)] > 0$ on $[a, b]$. Using (2) and (3), we obtain $\| W(\cdot, F) - W(\cdot, G)\| < \epsilon$ and $\sigma(-1)[W(x, F(x)) - W(x, G(x))] > 0$ on $[a, b]$. Thus, Definition 3′ is satisfied for $n_1 = 1$. If $n_1 > 1$, a similar argument using (2) and (3) holds. Therefore, $W(x, F(x))$ has (n_1, n_2) as a degree with respect to $\mathscr{W}$. Q.E.D.

We give next results on the uniqueness of best approximation with respect to a varisolvent family. Recall that the generalized exponential family E_n is varisolvent. In [2, p. 315], Braess has presented a class of continuous functions, each having at least two best approximations from E_2. Thus, best approximation in a varisolvent family is, in general, not unique.

From the class of functions given by Braess, it is easy to choose one with two best approximations from E_2, each best approximation having a degree (3, 4). This fact led to the following conjecture. Let $f \in C[a, b]$, let $\mathscr{F}$ be a varisolvent family, and define $\mathscr{F}_o = \{G \in \mathscr{F} \mid \text{if } (n_1, n_2) \text{ is a degree for } G, \text{ then } n_1 = 0\}$. Assume $F(x) \in \mathscr{F}$ is a best approximation to f with a degree (n_1, n_2) where $n_1 = n_2$. It was conjectured that $F(x)$ is then the unique approximation to f from $\mathscr{F} - \mathscr{F}_o$. We show with two examples that this conjecture is false. First, we state the following characterization theorem for best approximations [6].

THEOREM 1. *Let $\mathscr{F}$ be a varisolvent family on $[a, b]$ and let $f \in C[a, b]$. Assume $F \in \mathscr{F}$ has a degree (n_1, n_2) with respect to $\mathscr{F}$.*

1. *If $\|f - F\| \leqslant \|f - G\|$ for all G in $\mathscr{F}$, then either $f(x) - F(x)$ is a constant function or $f(x) - F(x)$ alternates n_1 times on $[a, b]$.*

2. *If $f(x) - F(x)$ alternates n_2 times on $[a, b]$ then $\|f - F\| \leqslant \|f - G\|$ for all $G \in \mathscr{F}$.*

Recall that the definition of "alternates" is as follows.

DEFINITION 5. Let $e \in C[a, b]$, e nonzero. Then $e(x)$ is said to alternate n times $(n \geqslant 0)$ on $[a, b]$ if there exist points $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$ such that $| e(x_i)| = \| e \|$ for all i $(1 \leqslant i \leqslant n + 1)$ and $e(x_i)\, e(x_{i+1}) < 0$ for all i $(1 \leqslant i < n + 1)$. The points $\{x_i\}_{i=1}^{n+1}$ are called an alternation set.

EXAMPLE 1. Let P_1 be the polynomials of degree one or less, let $[a, b] = [0, 5\pi/2]$, and let R denote the real numbers. Define $L \in C[0, 5\pi/2]$ as

$$\begin{aligned} L(x) &= -(2/\pi)\, x + 1, & 0 \leqslant x \leqslant \pi/2 \\ &= 0 & \pi/2 < x \leqslant 5\pi/2. \end{aligned}$$

It is easy to see that the family $\mathscr{F} = P_1 \cup \{L(x) + r \mid r \in R\}$ is varisolvent, where (2, 2) is a degree for $F \in \mathscr{F}$ if $F \in P_1$ and (1, 3) is a degree for $F \in \mathscr{F}$ if $F \in \{L(x) + r \mid r \in R\}$. Note that $\mathscr{F}_o$ is empty, and thus, $\mathscr{F} - \mathscr{F}_o = \mathscr{F}$.

Consider $f(x) = \sin x$. Observe that the zero function, $O(x)$, in P_1 has degree (2, 2) in $\mathscr{F}$ and that $f(x) - O(x)$ alternates twice on $[a, b]$. Thus, $O(x)$ is a best approximation to $f(x)$ from $\mathscr{F}$. However, $L(x)$ is also a best approximation to $f(x)$ from $\mathscr{F}$.

EXAMPLE 2. Let P_3 be the polynomials of degree three or less and let $[a, b] = [-1, 1]$. It has been shown in [6] that $\mathscr{F}_1 = \{O(x) \mid O(x)$ is the zero function in $P_3\} \cup \{F \in P_3 \mid$ for some x_1, x_2 with $-1 \leqslant x_1 < x_2 \leqslant 1$, $F(x_1)\, F(x_2) < 0\}$ is a varisolvent family with a degree (4, 4) for each $F \in \mathscr{F}_1$. It is easy to verify that $\mathscr{F} = \mathscr{F}_1 \cup \{2 \mid x \mid + r \mid r \in R\}$ is a varisolvent family with a degree (4, 4) for $F \in \mathscr{F}$ if $F \in \mathscr{F}_1$ and a degree (1, 5) for $F \in \mathscr{F}$ if $F(x) = 2 \mid x \mid + r$ for some $r \in R$. Note that $\mathscr{F}_o$ is empty, and thus, $\mathscr{F} - \mathscr{F}_0 = \mathscr{F}$.

Consider $f(x) \equiv 1$. The zero function from P_3 is a best approximation to $f(x)$ from $\mathscr{F}$. But $2 \mid x \mid$ is also a best approximation to $f(x)$ from $\mathscr{F}$.

In both examples, we have a best approximation $O(x) \in \mathscr{F}$ with a degree (n_1, n_2), $n_1 = n_2$, which is nonunique in $\mathscr{F} - \mathscr{F}_o$. In example 1, $f(x) - O(x)$ alternates, and in Example 2, $f(x) - O(x)$ is a constant function. However, the following uniqueness result does hold.

THEOREM 2. *Let $f \in C[a, b]$ and let $F(x)$ be a best approximation to f on $[a, b]$ from a varisolvent family $\mathscr{F}$. Assume F has a degree (n_1, n_2) with $n_1 = n_2$ and that $f(x) - F(x)$ alternates n_1 times. Then if $G \in \mathscr{F}$ is a best approximation to f from $\mathscr{F}$ with a degree (m_1, m_2), $m_1 \geqslant 1$, then $f(x) - G(x)$ must alternate m_1 times. (In particular, $f(x) - G(x)$ may not be a constant function.)*

Proof. We show that it is impossible for $f(x) - G(x)$ to be a constant function. The theorem then follows from part 1 of Theorem 1.

Assume $f(x) - G(x)$ is a constant function, i.e., $f(x) - G(x) \equiv C$ on $[a, b]$, C a real number. Since F is a best approximation to f and $f(x) - F(x)$ alternates $n_1 \geqslant 1$ times, we know $C \neq 0$. Assume $C > 0$. (A similar argument holds if $C < 0$.). Note that $\|f - F\| = C$.

Case 1. $m_1 = 1$

By varisolvency, there exists a $G_1 \in \mathscr{F}$ with $\| G_1 - G \| < C$ and with $G_1(x) - G(x) > 0$ on $[a, b]$. Hence, we have $\| f - G_1 \| < C$. This contradicts the fact that $G(x)$ is a best approximation to $f(x)$.

Observe that if $G(x)$ has $(1, m_2)$ as a degree, where $m_1 > 1$, the previous argument holds. Cases 2 and 3 handle the situation that $m_1 > 1$, and that $G(x)$ does not have $(1, m_2)$ as a degree.

Case 2. $m_1 = 2k + 1; k \geqslant 1$

Let $A = \{x_i\}_{i=1}^{n_1+1}$ be a set of alternation points for $f(x) - F(x)$. Define $x_j = \min_{1 \leqslant i \leqslant n_1+1} \{x_i \in A \mid F(x_i) = f(x_i) + C\}$.

Subcase 2a. $a < x_j < b$. By continuity, there exists an $\alpha > 0$ such that $\alpha < \min\{x_j - a, b - x_j\}$ and such that for all $x \in I_j \doteq (x_j - \alpha, x_j + \alpha)$, $F(x) > f(x)$. Let $\beta = (2\alpha)/[(m_1 - 2) + (m_1 - 2) + 1] > 0$. Consider the following increasing sequence of closed intervals: $[c_1, d_1] = [a, x_j - \alpha]$, $[c_{m_1}, d_{m_1}] = [x_j + \alpha, b]$ and

$$[c_i, d_i] = [x_j - \alpha + (2i - 3)\,\beta, x_j - \alpha + 2(i - 1)\,\beta]$$

for all i $(2 \leqslant i \leqslant m_1 - 1)$. The varisolvency of $G(x)$ guarantees the existence of $G_1 \in \mathscr{F}$ with $\| G_1 - G \| < C/2$ and $(-1)(-1)^i[G_1(x) - G(x)] > 0$, on $[c_i, d_i]$ for all i $(1 \leqslant i \leqslant m_1)$. Observe that $G_1(x) > G(x)$ on $[a, b] - I_j$ and that $F(x) > f(x)$ on I_j. Since $A = \{x_i\}_{i=1}^{n_1+1}$ is an alternation set for $f(x) - F(x)$, it follows that $[G_1(x_i) - F(x_i)][G_1(x_{i+1}) - F(x_{i+1})] < 0$ for all i $(1 \leqslant i \leqslant n_1)$. Thus, $G_1(x) - F(x)$ has $n_1 = n_2$ sign changes on $[a, b]$, a contradiction.

Subcase 2b. $x_j = a$. We modify the proof of Subcase 2a. By continuity, there exists an $\alpha > 0$ such that $\alpha < b - a$ and such that for all $x \in I_j \doteq [x_j, x_j + \alpha)$, $F(x) > f(x)$. Let $\beta = \alpha/[(m_1 - 1) + (m_1 - 1)] > 0$. Consider the following increasing sequence of closed intervals:

$$[c_i, d_i] = [x_j + 2(i - 1)\,\beta, x_j + (2i - 1)\,\beta] \qquad (1 \leqslant i \leqslant m_1 - 1),$$

and $[c_{m_1}, d_{m_1}] = [x_j + \alpha, b]$. The varisolvency of $G(x)$ guarantees the existence of $G_1 \in \mathscr{F}$ such that

$$\| G_1 - G \| < C/2 \qquad \text{and} \qquad (-1)(-1)^i[G_1(x) - G(x)] > 0$$

on $[c_i, d_i]$ for all i $(1 \leqslant i \leqslant m_1)$. Observe that $G_1(x) > G(x)$ on $[a, b] - I_j =$

$[x_j + \alpha, b]$ and that $F(x) > f(x)$ on I_j. Again since $\{x_i\}_{i=1}^{n_1+1}$ is an alternation set for $f(x) - F(x)$, it follows that

$$[G_1(x_i) - F(x_i)][G_1(x_{i+1}) - F(x_{i+1})] < 0 \qquad \text{for all } i \ (1 \leqslant i \leqslant n_1).$$

Hence, $G_1(x) - F(x)$ has $n_1 = n_2$ sign changes on $[a, b]$, a contradiction.

Subcase 2c. $x_j = b$. This is handled by a proof similar to Subcase 2b.

Case 3. $m_1 = 2k, k \geqslant 2$

It is shown in [6] that if $G \in \mathscr{F}$ has a degree (m_1, m_2), then $(m_1 - 1, m_2)$ is also a degree with respect to $\mathscr{F}$ as long as m_1 is not zero or three. Hence, Case 3 reduces to Case 2. Q.E.D.

3. Generalized Rational Approximation

The definition of generalized rational functions given by Cheney in [4] is as follows. Let P and Q denote two finite-dimensional subspaces of $C[a, b]$. It is assumed that Q contains at least one function that is positive throughout $[a, b]$. The approximating family

$$R^* = \{p(x)/q(x) \mid p \in P, q \in Q, x \in [a, b], q(x) > 0 \text{ on } [a, b]\}$$

is called a family of generalized rational functions. Henceforth, R^* will denote an aribtrary nonempty family of generalized rational functions.

We give first Rice's definition of varisolvency and then show R^* is not, in general, varisolvent with respect to Rice's definition. Recall that in special cases, for example in the case R^* is a family of rational polynomial functions, R^* is varisolvent with respect to Rice's definition, (cf. [10, p. 78]).

Definition 6. Let $\mathscr{F}$ be a family of functions in $C[a, b]$ and let $F \in \mathscr{F}$. Then F is said to have the integer $n \geqslant 1$ as a degree with respect to $\mathscr{F}$ if the following conditions hold:

1. Let an arbitrary set of n distinct points $\{x_i\}_{i=1}^n$ in $[a, b]$ and let $\epsilon > 0$ be given. Then there exists a $\delta(F, \epsilon, \{x_i\}_{i=1}^n) = \delta > 0$ such that for any set of real numbers $\{y_i\}_{i=1}^n$ with $|y_i - F(x_i)| < \delta$ for all i $(1 \leqslant i \leqslant n)$, there exists a $G \in \mathscr{F}$ with $G(x_i) = y_i$ for all i $(1 \leqslant i \leqslant n)$ and $\| F - G \| < \epsilon$.

2. If G is in $\mathscr{F}$ with $G(x_i) = F(x_i)$ for all i $(1 \leqslant i \leqslant n)$, where $\{x_i\}_{i=1}^n$ is a set of n distinct points in $[a, b]$, then F and G are identical.

Definition 7. Let $\mathscr{F}$ be a nonempty family of functions in $C[a, b]$. Then $\mathscr{F}$ will be called a varisolvent (Rice) family if every $F \in \mathscr{F}$ has a degree with respect to $\mathscr{F}$. Example 3 shows that R^* is not, in general, a varisolvent (Rice) family.

EXAMPLE 3. Let $[a, b] = [-1, 1]$, let Q be the linear space of constant functions on $[a, b]$ and let P be the linear span of x. Consider the generalized rational family $R^* = \{p(x)/q(x) \mid p \in P,\ q \in Q,\ q(x) > 0 \text{ on } [-1, 1]\}$. We show it is impossible to assign a degree to the zero function, $O(x)$, in R^*.

Assume R^* is varisolvent (Rice) on $[-1, 1]$ and that $O(x)$ has a degree $n \geqslant 1$. Since $r(x) = x \in R^*$ has one zero with $O(x)$, this implies $n \geqslant 2$. Consider

$$\begin{aligned} f(x) &= 1, & -1 \leqslant x \leqslant 0 \\ &= -2x + 1, & 0 < x \leqslant 1. \end{aligned}$$

$O(x)$ is a best approximation to $f(x)$ from R^*. However, $f(x) - O(x)$ alternates only once. Since $n \geqslant 2$, this contradicts the following characterization theorem (cf. [8, 11]).

THEOREM 3. *Let $\mathscr{F}$ be a varisolvent (Rice) family, let $f \in C[a, b]$ and let $F \in \mathscr{F}$ have degree n. Then F is a best approximation to f from $\mathscr{F}$ on $[a, b]$ iff either*

1. *$f(x) - F(x)$ alternates n times on $[a, b]$, or*
2. *$f(x) - F(x)$ is a constant function on $[a, b]$.*

We next proceed to the proof that a family of generalized rational functions is varisolvent in the extended sense of Gillotte and McLaughlin. The following notation is used. Let M be a finite-dimensional subspace of $C[a, b]$. Then

$\delta(M) =$ dimension of M,

$\gamma(M) = 1 +$ maximum number of variations in sign possessed by members of M, ($\gamma(M) = +\infty$ possible),

$\eta(M) = \max\{\delta(H) \mid H \text{ is a Haar subspace of } M\}$.

Recall that an n-dimensional subspace H of $C[a, b]$ is an n-dimensional Haar subspace if each nonzero element of H has at most $n - 1$ distinct zeros in $[a, b]$. For a fixed element $r(x) = p(x)/q(x) \in R^*$, define

$$P + rQ = \{p(x) + r(x)\, q(x) \mid p \in P,\ q \in Q,\ x \in [a, b]\}.$$

Note that $P + rQ$ is a linear subspace of $C[a, b]$.

THEOREM 4. *Let R^* be a family of generalized rational functions. Then R^* is a varisolvent family where an element $r \in R^*$ has a degree (n_1, n_2) with $n_1 = \eta(P + rQ)$ and $n_2 = \gamma(P + rQ)$.*

Proof. Let $r(x) = p(x)/q(x)$ be an arbitrary element of R^* and let $n_1 = \eta(P + rQ)$, $n_2 = \gamma(P + rQ)$. Consider first $n_2 = \gamma(P + rQ)$. If $n_2 = +\infty$, there is nothing to show. Thus, suppose $n_2 < +\infty$ and assume there exists an $r_1(x) = p_1(x)/q_1(x) \in R^*$ such that $r_1(x) - r(x)$ has n_2 sign changes on $[a, b]$. Hence, there exists points

$$a \leqslant x_1 < \cdots < x_{n_2+1} \leqslant b$$

such that

$$[r_1(x_i) - r(x_i)][r_1(x_{i+1}) - r(x_{i+1})] < 0, \qquad \text{for all } i\ (1 \leqslant i \leqslant n_2).$$

Since $q_1(x) > 0$ on $[a, b]$, we have

$$[p_1(x_i) - r(x_i)\, q_1(x_i)][p_1(x_{i+1}) - r(x_{i+1})\, q_1(x_{i+1})] < 0$$

for all i $(1 \leqslant i \leqslant n_2)$. Thus, $p_1(x) - r(x)\, q_1(x)$ has $n_2 = \gamma(P + rQ)$ sign changes on $[a, b]$. But $p_1(x) - r(x)\, q_1(x) = p_1(x) + r(x)[-q_1(x)]$ is an element of $P + rQ$. This contradicts the definition of $\gamma(P + rQ)$.

Consider next $n_1 = \eta(P + rQ)$. If $n_1 = \eta(P + rQ) = 0$, there is nothing to show. Thus, assume $n_1 \geqslant 1$. We use the following remark given in [6]. Suppose K is an n_1-dimensional Haar subspace on $[a, b]$. Let δ be an arbitrary element of $\{0, 1\}$ with $\delta < n_1$. Let $\{x_i\}_{i=1}^{n_1+1-\delta}$ be a set of points with $a = x_1 < \cdots < x_{n_1+1-\delta} = b$. Then there exists a $k \in K$ with $k(a)\, k(b) \neq 0$ such that $(-1)^i\, k(x) > 0$ on (x_i, x_{i+1}) for all i $(1 \leqslant i \leqslant n_1 - \delta)$.

Consider part 1 of Definition 3. Let $\epsilon > 0$ and σ in $\{-1, 1\}$ be arbitrarily chosen. Let δ be an aribitrary element of $\{0, 1\}$ with $\delta < n_1$. Assume that $\{[c_i, d_i]\}_{i=1}^{n_1-\delta}$ is an arbitrary increasing sequence of closed intervals with $c_1 = a$ and $d_{n_1-\delta} = b$. Define $x_1 = a$, $x_i = (1/2)(d_{i-1} + c_i)$ for all i $(2 \leqslant i \leqslant n_1 - \delta)$ and $x_{n_1-\delta+1} = b$. By the definition of $n_1 = \eta(P + rQ)$ and the previous remark, we know there exists a function $k \in P + rQ$, with $k(a)\, k(b) \neq 0$ such that $(-1)^i\, k(x) > 0$ on (x_i, x_{i+1}) for all i $(1 \leqslant i \leqslant n_1 - \delta)$. Using the continuity of $k(x)$ one can actually show that if $n_1 - \delta > 1$, $(-1)\, k(x) > 0$ on $[x_1, x_2)$, $(-1)^{n_1-\delta}\, k(x) > 0$ on $(x_{n_1-\delta}, x_{n_1-\delta+1}]$ and if $n_1 - \delta = 1$, $(-1)\, k(x) > 0$ on $[x_1, x_2]$. Since $[c_1, d_1] \subset [x_1, x_2)$, $[c_i, d_i] \subset (x_i, x_{i+1})$ for all i $(1 < i < n_1 - \delta)$ and $[c_{n_1-\delta}, d_{n_1-\delta}] \subset (x_{n_1-\delta}, x_{n_1-\delta+1}]$ for $n_1 - \delta > 1$ and $[c_1, d_1] \subset [x_1, x_2]$ for $n_1 - \delta = 1$, we have $(-1)^i\, k(x) > 0$ on $[c_i, d_i]$ for all i $(1 \leqslant i \leqslant n_1 - \delta)$.

Let $\alpha(x) = \sigma k(x)$. Since $\alpha \in P + rQ$, $\alpha(x)$ may be written as $\alpha(x) = p_1(x) + (p(x)/q(x))\, q_1(x)$ for some $p_1 \in P$ and $q_1 \in Q$. This implies

$$p_1(x)\, q(x) + p(x)\, q_1(x) = q(x)\, \alpha(x). \tag{4}$$

Consider

$$r_1(x) \doteq \frac{p(x) - \epsilon_1 p_1(x)}{q(x) + \epsilon_1 q_1(x)},$$

where $\epsilon_1 > 0$, ϵ_1 to be chosen later, is sufficiently small so that $q(x) + \epsilon_1 q_1(x) > 0$ on $[a, b]$. Since P and Q are subspaces and $q(x) + \epsilon_1 q_1(x) > 0$ on $[a, b]$, r_1 is in R^*. Using (4), we obtain

$$r(x) - r_1(x) = \frac{p(x)}{q(x)} - \frac{p(x) - \epsilon_1 p_1(x)}{q(x) + \epsilon_1 q_1(x)} = \frac{\epsilon_1(p_1(x)\, q(x) + p(x)\, q_1(x))}{q(x)(q(x) + \epsilon_1 q_1(x))}$$
$$= \frac{\epsilon_1 q(x)\, \alpha(x)}{q(x)(q(x) + \epsilon_1 q_1(x))} = \frac{\epsilon_1 \sigma k(x)}{q(x) + \epsilon_1 q_1(x)}.$$

Thus

$$\begin{aligned}\sigma(-1)^i[r(x) - r_1(x)] &= \sigma(-1)^i[(\epsilon_1 \sigma k(x))/(q(x) + \epsilon_1 q_1(x))] \\ &= [\epsilon_1/(q(x) + \epsilon_1 q(x))](-1)^i\, k(x) > 0, \\ &\text{on } [c_i, d_i] \text{ for all } i\ (1 \leqslant i \leqslant n_1 - \delta).\end{aligned}$$

We note that $\| r - r_1 \| = [\max_{x\in[a,b]} (| k(x)|/| q(x) + \epsilon_1 q_1(x)|)] \cdot \epsilon_1$. Since $q(x) > 0$ on $[a, b]$, a short argument shows ϵ_1 can be chosen sufficiently small such that $\| r - r_1 \| < \epsilon$ and $q(x) + \epsilon_1 q_1(x) > 0$ on $[a, b]$ hold simultaneously. Hence, $r_1 \in R^*$ has the required properties stated in Definition 3, part 1. Therefore, (n_1, n_2) with $n_1 = \eta(P + rQ)$ and $n_2 = \gamma(P + rQ)$ is a degree for $r \in R^*$. Q.E.D.

LEMMA 3. *Let R^* be a family of generalized rational functions. Let $r \in R^*$ have a degree (n_1, n_2), where $n_1 = \eta(P + rQ)$ and $n_2 = \gamma(P + rQ)$. If $\eta(P + rQ) \geqslant 1$, then r has $(1, n_2)$ as a degree with respect to R^*.*

Proof. By definition, $\eta(P + rQ) \geqslant 1$ means that $P + rQ$ contains a Haar subspace of dimension greater than or equal to one. Recall that every Haar subspace of dimension $n_1 \geqslant 1$ contains a function that is positive on $[a, b]$ (cf. [1]). Thus, there exists a function $k \in P + rQ$ such that $(-1)\, k(x) > 0$ on $[a, b]$. The proof is completed by employing the arguments that appear in the proof of Theorem 4. Q.E.D.

Remark 1. Let F be in $\mathscr{F}$ where $\mathscr{F}$ is a varisolvent family. Assume F has $(1, n_2)$ as a degree and that F is a best approximation to $f \in C[a, b]$ from $\mathscr{F}$. It is easy to see that $f(x) - F(x)$ may not be a nonzero constant function. Thus, Lemma 3 implies that if r is a best approximation to $f \in C[a, b]$ from R^* and $\eta(P + rQ) \geqslant 1$, $f(x) - r(x)$ is not a nonzero constant function.

A number of results concerning a family of generalized rational functions now follow from the fact that such a family is varisolvent. We state first the characterization theorem for best approximations [4]. This theorem uses the following modified definition of alternation.

DEFINITION 8. A function $e \in C[a, b]$ is said to alternate $n \geqslant 0$ times on $[a, b]$ if exist points $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$ such that $e(x_i) = (-1)^i \lambda$ for all i $(1 \leqslant i \leqslant n + 1)$ with $| \lambda | = \| e \|$.

THEOREM 5. *Let R^* be a family of generalized rational functions and let $r \in R^*$. If the error function $e = f - r$ alternates at least $\gamma(P + rQ)$ times, then r is a best approximation to f from R^*. If r is a best approximation to f from R^*, then e alternates at least $\eta(P + rQ)$ times.*

We see from Theorem 4 and Remark 4 that Theorem 5 follows from the general characterization theorem for varisolvent families, i.e., Theorem 1.

We note that in the situation $r \in R^*$ is a best approximation and $n_1 = \eta(P + rQ) = 0$, Theorem 5 allows the possibility that $f(x) - r(x)$ is a nonzero constant function. Example 3 illustrates that this can occur. Recall that in Example 3, $[a, b] = [-1, 1]$, Q is the constant functions and P is the linear span of x. Each $r \in R^*$ has $(0, 2)$ as a degree. Observe that the zero function $O(x)$, is a best approximation to $f(x) \doteq 1$ from R^* and that $e(x) = f(x) - O(x)$ is a nonzero constant function.

We mention next the de La Vallée Poussin theorem for generalized rational functions (cf. [4, p. 163]).

THEOREM 6. *Let R^* be a family of generalized rational functions, let $r \in R^*$ and let $f \in C[a, b]$. Assume that $f - r$ is alternately positive and negative at the points $a \leqslant x_1 < \cdots < x_k \leqslant b$ with $k > \gamma(P + rQ)$. Then* $\inf_{r_1 \in R^*} \|f - r_1\| \geqslant \min_{1 \leqslant i \leqslant k} |f(x_i) - r(x_i)|$.

This result now follows from Lemma 1.

Finally we mention Lemma 2. If R^* is a family of generalized rational functions, Lemma 2 implies that $\mathscr{W} = \{W(x, r(x)) \mid r \in R^*, x \in [a, b]\}$ is a varisolvent family. In particular, the characterization theorem (Theorem 1) for varisolvent families may be applied to the problem of approximating $f \in C[a, b]$ by elements of $\mathscr{W}$. A result of this type has been given by Moursund and Taylor. They use the following terminology. Assume that $W(x, y)$ satisfies: (a) $W(x, y)$ is continuous on $[a, b] \times (-\infty, \infty)$, (b) $\operatorname{sgn} W(x, y) = \operatorname{sgn} y$ for all x in $[a, b]$ and (c) for each x, $W(x, y)$ is strictly monotone increasing in y with $\lim_{|y| \to \infty} |W(x, y)| = \infty$. For fixed $f \in C[a, b]$, $f \notin R^*$ and $r \in R$, the weighted error curve $W(x, f(x) - r(x))$ is said to alternate $n \geqslant 0$ times if there exist points $a \leqslant x_1 < \cdots < x_{n+1} \leqslant b$ such that $|W(x_i, f(x_i) - r(x_i))| = (-1)^i \lambda$, for all i $(1 \leqslant i \leqslant n + 1)$, where $|\lambda| = \|W(\cdot, f - r)\|$. The problem under consideration is: given $f \in C[a, b], f \notin R^*$, find $r \in R^*$ such that $\|W(\cdot, f - r)\| \leqslant \inf_{r_1 \in R^*} \|W(\cdot, f - r_1)\|$. Such an r is called a best approximation to f "with respect to the generalized weight function, $W(x, y)$."

The following result appears in [9].

THEOREM 7. *Let $f \in C[a, b]$, $f \notin R^*$ and let $r \in R^*$, where R^* is a family of generalized rational functions. If $W(x, f(x) - r(x))$ alternates at least*

$\gamma(P + rQ)$ times, then r is a best approximation to f with respect to $W(x, y)$. If r is a best approximation to f with respect to $W(x, y)$ then $W(x, f(x) - r(x))$ alternates at least $\eta(P + rQ)$ times.

Remark 2. For a given $f \in C[a, b]$ and a given family R^*, it is easy to show $\mathscr{W}_1 = \{W(x, f(x) - r(x)) | r \in R^*\}$ is a varisolvent family. In particular, if $r \in R^*$ has a degree (n_1, n_2), then (n_1, n_2) is also a degree for $W(x, f(x) - r(x))$. The proof that $\mathscr{W}_1$ is varisolvent follows the same reasoning that is used in the proof of Lemma 2.

Remark 3. For a given $f \in C[a, b]$ and a given family R^*, the following two problems are equivalent.

Problem 1. Find a best approximation $r \in R^*$ to f with respect to $W(x, y)$.

Problem 2. Find a best approximation $W(x, f(x) - r(x)) \in \mathscr{W}_1$ to the zero function on $[a, b]$.

We observe that from Remarks 2 and 3, Theorem 7 follows from the general characterization theorem for varisolvent families, Theorem 1.

Some results on generalized rational functions do not follow, however, from varisolvency. We mention the uniqueness theorem that appears in [3, p. 104].

THEOREM 8. *Let R^* be a family of generalized rational functions, let $f \in C[a, b]$, and let $r \in R^*$ be a best approximation to f. If $P + rQ$ is a Haar subspace, then r is unique.*

Remark 4. Assume $r \in R^*$, with $P + rQ$ a Haar subspace, is a best approximation to f. We show one cannot conclude that r is unique from the fact R^* is a varisolvent family. Recall that r is a varisolvent function with a degree (n_1, n_2); $n_1 = \eta(P + rQ)$, and $n_2 = \gamma(P + rQ)$.

We use two facts given in [3]. Fact one is: $\gamma(P + rQ) \geqslant \delta(P + rQ) \geqslant \eta(P + rQ)$ holds for any $r \in R^*$, and fact two is: $P + rQ$ is a Haar subspace if and only if $\delta(P + rQ) = \eta(P + rQ)$.

Assume now that $n_1 = n_2$. This implies that $\delta(P + rQ) = \eta(P + rQ)$. Hence, $P + rQ$ is a Haar subspace. Theorem 8 quarantees that r is unique. However, Examples 1 and 2 show that there exist varisolvent families with a best approximation, $F(x)$, having a degree (n_1, n_2), $n_1 = n_2$, and yet $F(x)$ is not unique.

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